数理解析研究所講究録
第 1602 巻 2008 年 93-102

On the Amalgamation Property for Automorphisms

神戸大学・工　桔梗宏孝 (Hirotaka Kikyo)
Graduate School of Engineering, Kobe University
kikyo@kobe-u.ac.jp

1 Introduction

The author found that if a model complete theory T is unstable and has the amalgamation property for automorphisms (PAPA) then T with the axiom scheme saying that "σ is an automorphism" has no model companion. This proposition still holds if T has the PAPA after adding enough constants to the language.

In this paper, we review some known examples which has the PAPA, and examples without the PAPA. Many theories without the PAPA will have the PAPA after adding enough constants to the language. Hrushovski found an theory whose expansion by any set of constants can never have the PAPA [2]. His example is supersimple with SU-rank 1. With some modification, we found a theory of a binary graph with the strict order property whose expansion by a set of constants can never have the PAPA.

We give some notational conventions. Capital letters like A and M usually denote structures, but X and Y are used for sets. We often write XY for $X \cup Y$. If A is a structure, we sometimes use $|A|$ for their domain in case we need to distinguish a structure and its domain. The cardinality of a set X will be denoted by $\#X$. If A is a structure and $X \subset |A|$, $A|X$ denotes the substructure of A with domain X. If a is a tuple of elements from A, we also write $a \in A$. For the sake of simplicity, we treat relational structures.

2 Quasi-Generic Structure

Definition 2.1. Let $\mathcal{L}$ be a relational language. Let $\mathbf{K}$ be an infinite class of finite structures for $\mathcal{L}$. $\mathbf{K}$ has the *quasi-hereditary property* (qHP for short) with a map $f : \omega \to \omega$ if $B \subset A \in \mathbf{K}$ then there is $B' \in \mathbf{K}$ such that $B \subset B' \subset A$ and $\#B' \leq f(\#B)$.

Theorem 2.2. *Let $\mathcal{L}$ be a relational language. We assume that for each positive integer n, there are only finitely many isomorphism types of the structures of size n in $\mathbf{K}$.*

Let $\mathbf{K}$ be an infinite class of finite $\mathcal{L}$-structures which has qHP with $f : \omega \to \omega$, JEP and AP. Then there is an $\mathcal{L}$-structure D, unique up to isomorphism such that

(1) *D is countable,*

(2) *for any finite substructure A of D there is $B \in \mathbf{K}$ such that $A \subset B \subset D$ with $\#B \leq f(\#A)$,*

(3) *whenever $A \subset D$ and $A \subset B$ with $A, B \in \mathbf{K}$ then there is an $\mathcal{L}$-substructure B' of D such that $A \subset B'$ and $B' \cong_A B$.*

Let T be a theory in $\mathcal{L}$ expressing (2) and (3). Then T is ω-categorical and model-complete.

Moreover, if A and A' are substructures of D with $A \in \mathbf{K}$ and $\sigma : A \to A'$ is an $\mathcal{L}$-isomorphism, then σ can be extended to an $\mathcal{L}$-automorphism of D.

Proof. The proof is the same as that for the existence of "$(\mathbf{K}, \leq)$-generic structure" (e.g., [8]). See the next remark.

We just prove the model-completeness of T. Let $M \subset M'$ be models of T. We show that M is existentially closed in M'. Let $\varphi(x, y)$ be a quantifier-free formula in $\mathcal{L}$ with x, y tuples of free variables, and suppose $M' \models \varphi(a', b)$ with a tuple $b \in M$ and a tuple $a' \in M' - M$. Choose a finite subset $Y \subset |M|$ such that $b \in Y$ and $M|Y \in \mathbf{K}$. Choose finite subset $X' \subset |M'|$ such that $a' \in X'$, $Y \subset X'$, and $M'|X' \in \mathbf{K}$. By (3) of T, there is $X \subset |M|$ such that $Y \subset X$ and $M'|X' \cong_Y M|X$. Let a be a tuple in X corresponding to a' through this isomorphism over Y. Then $M \models \varphi(a, b)$. □

Definition 2.3. A class of finite structures $\mathbf{K}$ is called a *quasi-generic class* if $\mathbf{K}$ satisfies the hypothesis of Theorem 2.2, and structure D in the theorem is called the *quasi-generic structure* of $\mathbf{K}$.

Remark 2.4. *Suppose $\mathbf{K}$ satisfies the hypothesis of Theorem 2.2. Then we can assume that $\mathbf{K}$ is closed under intersections. Let*

$$\overline{\mathbf{K}} = \{A : A \subset {}^{\exists}B \in \mathbf{K}\}.$$

We can consider $\mathbf{K}$ as a set of closed structures in $\overline{\mathbf{K}}$. If $\mathbf{K} = \overline{\mathbf{K}}$, the quasi-generic structure of $\mathbf{K}$ is the usual generic structure of $\mathbf{K}$.

3 Theories with the PAPA

If σ is an automorphism of M and $f : M \to M'$ is an isomorphism, we let $\sigma^f = f \circ \sigma \circ f^{-1}$. We write

$$f : M \underset{A}{\to} M'$$

if $A \subset M, M'$ and f is a map from M to M' such that $f(a) = a$ for each $a \in A$.

Definition 3.1 (Lascar[6]). Suppose M_0, M_1, M_2 are models of T with $M_0 \prec M_1, M_2$, and σ_0, σ_1, σ_2 automorphisms of M_0, M_1, and M_2 respectively with $\sigma_1|M_0 = \sigma_2|M_0 = \sigma_0$. We say that (M_1, σ_1) and (M_2, σ_2) can be amalgamated over (M_0, σ_0) if there is an elementary extension M_3 of M_0 and an automorphism σ_3 of M_3 such that there are elementary embeddings

$$f_1 : M_1 \xrightarrow[M_0]{\prec} M_3, \quad f_2 : M_2 \xrightarrow[M_0]{\prec} M_3$$

over M_0 such that $\sigma_3 | f_1(M_1) = \sigma_1^{f_1}$ and $\sigma_3 | f_1(M_1) = \sigma_1^{f_2}$. A complete theory T has *the amalgamation property for automorphisms* (PAPA) if whenever M_0, M_1, M_2 are models of T with $M_0 \prec M_1, M_2$ and σ_0, σ_1, σ_2 automorphisms of M_0, M_1, and M_2 respectively with $\sigma_1 | M_0 = \sigma_2 | M_0 = \sigma_0$ then (M_1, σ_1) and (M_2, σ_2) can be amalgamated over (M_0, σ_0).

Definition 3.2. Let $\mathcal{L}$ be a relational language. Suppose A, B, C are $\mathcal{L}$-structures and $A \subset B$, $A \subset C$. An $\mathcal{L}$-structure D is a *free amalgam* of B and C over A if there are $\mathcal{L}$-isomorphisms $f_1 : B \to D$ and $f_2 : C \to D$ such that $|D| = f_1(|B|) \cup f_2(|C|)$, $D^R = D|f_1(|B|)^R \cup D|f_2(|C|)^R$ for each relation R in $\mathcal{L}$, $f_1(|B|) \cap f_2(|C|) = f_1(|A|) = f_2(|A|)$ and $f_1(x) = f_2(x)$ for each $x \in |A|$.

Theorem 3.3. *Suppose* $\mathbf{K}$ *satisfies the conditions of Theorem 2.2, and closed under unions and intersections in the following sense: For each structure* $D \in \mathbf{K}$, *whenever* $D|X$ *and* $D|Y$ *are members of* $\mathbf{K}$ *with* $X, Y \subset |D|$, *then* $D|(X \cap Y)$ *and* $D|(X \cup Y)$ *are members of* $\mathbf{K}$. *If* $\mathbf{K}$ *is closed under free amalgams then the theory of the quasi-generic structure of* $\mathbf{K}$ *has the PAPA.*

Proof. Let T be the theory of the quasi-generic structure of $\mathbf{K}$. Suppose M_0, M_1, M_2 are models of T such that $M_0 \prec M_1$, $M_0 \prec M_2$, and σ_0, σ_1, σ_2 automorphisms of M_0, M_1, M_2 respectively and $\sigma_i | M_0 = \sigma_0$ for $i = 1, 2$. By renaming, we can assume that $M_1 \cap M_2 = M_0$.

Claim 1. *With* $X = |M_1|$, $Y = |M_2|$, *there is a model* $M \models T$ *and a map* $\sigma : M \to M$ *such that*

(1) $X \cup Y \subset M$,

(2) $\sigma(x) = \begin{cases} \sigma_1(x) & (x \in X) \\ \sigma_2(x) & (x \in Y), \end{cases}$

(3) $M|X = M_1$,

(4) $M|Y = M_2$, *and*

(5) σ *is an automorphism of* M.

By compactness, it is enough to show the following claim:

Claim 2. *For each finite subsets* $X \subset |M_1|$ *and* $Y \subset |M_2|$ *such that* $X \cap Y = X \cap M_0 = Y \cap M_0$, *there is a* $M \models T$ *such that*

(1) $X X^{\sigma_1} Y Y^{\sigma_2} \subset M$,

(2) $\sigma(x) = \begin{cases} \sigma_1(x) & (x \in X) \\ \sigma_2(x) & (x \in Y), \end{cases}$

(3) $M|(|M_1| \cap (XX^{\sigma_1}YY^{\sigma_2})) = M_1|(|M_1| \cap (XX^{\sigma_1}YY^{\sigma_2}))$,

(4) $M|(|M_2| \cap (XX^{\sigma_1}YY^{\sigma_2})) = M_2|(|M_2| \cap (XX^{\sigma_1}YY^{\sigma_2}))$, *and*

(5) σ *is an automorphism of* M.

Let $X_1 \subset |M_1|$ and $Y_1 \subset |M_2|$ be finite subsets such that $X_1 \cap Y_1 = X_1 \cap M_0 = Y_1 \cap M_0$.

Choose finite sets X_2 and Y_2 such that

$$X \subset X_2 \subset |M_1|,\ M_1|X_2 \in \mathbf{K},$$
$$Y \subset Y_2 \subset |M_2|, \text{ and } M_2|Y_2 \in \mathbf{K}.$$

Since M_0 is a model of T, $M_0 \subset M_1$ and $\mathbf{K}$ is closed under intersections, $M_0|(X_2 \cap |M_0|) \in \mathbf{K}$. Similarly, $M_0|(Y_2 \cap |M_0|) \in \mathbf{K}$. Since $\mathbf{K}$ is closed under unions, for $X = X_2 \cup (Y_2 \cap |M_0|)$ and $Y = Y_2 \cup (X_2 \cap |M_0|)$, $M_1|X$, $M_2|Y$ and $M_0|X \cap Y$ are members of $\mathbf{K}$. Note that $X \cap Y = X \cap |M_0| = Y \cap |M_0|$.

By assumption on σ_1 and σ_2, $M_1|X^{\sigma_1}$, $M_2|Y^{\sigma_2}$, and $M_0|X^{\sigma_1} \cap Y^{\sigma_2}$ are also members of $\mathbf{K}$. Since $\mathbf{K}$ is closed under unions, $M_1|XX^{\sigma_1}$, $M_2|YY^{\sigma_2}$, and $M_0|(XX^{\sigma_1}) \cap (YY^{\sigma_2})$ are members of $\mathbf{K}$. Let D be the free amalgam of $M_1|XX^{\sigma_1}$ and $M_2|YY^{\sigma_2}$ over $M_0|(XX^{\sigma_1}) \cap (YY^{\sigma_2})$. Then $D \in \mathbf{K}$. By inspection, $D|XY$ is a free amalgam of $M_1|X$ and $M_2|Y$ over $M_0|X \cap Y$, thus $D|XY \in \mathbf{K}$. Similarly, $D|X^{\sigma_1}Y^{\sigma_2}$ is a free amalgam of $M_1|X^{\sigma_1}$ and $M_2|Y^{\sigma_2}$ over $M_0|X^{\sigma_1} \cap Y^{\sigma_2}$, and $D|X^{\sigma_1}Y^{\sigma_2} \in \mathbf{K}$. Let σ be a map from $D|XY$ to $D|X^{\sigma_1}Y^{\sigma_2}$ such that $\sigma(x) = \sigma_1(x)$ if $x \in X$ and $\sigma(x) = \sigma_2(x)$ if $x \in Y$. Since $D|XY$ and $D|X^{\sigma_1}Y^{\sigma_2}$ are free amalgams, σ is an isomorphism. Let M be a countable model of T such that $D \subset M$. By Theorem 2.2, σ can be extended to an automorphism of M. Hence, we have Claim 2. □

Fact 3.4. *The theory of the dense linear order without endpoints has the PAPA.*

Proof. Let $(M_1, <_1)$, $(M_2, <_2)$ be dense linear orders without end points such that $M_0 \subset M_1$, $M_0 \subset M_2$ and $(M_0, <_1) = (M_0, <_2)$.

By renaming elements of M_1 and M_2, we can assume that $M_1 \cap M_2 = M_0$. Let $\sigma = \sigma_1 \cup \sigma_2$. Then it is well-defined on $M_1 \cup M_2$.

We define a binary relation $<_3$ on $M_1 \cup M_2$ as follows:

$x <_3 y$ if and only if

(1) $x <_1 y$ with $x, y \in M_1$;

(2) $x <_2 y$ with $x, y \in M_2$;

(3) $x \in M_2 - M_0$, $y \in M_1 - M_0$ and $x <_2 a <_1 y$ for some $a \in M_0$; or

(4) $x \in M_1 - M_0$, $y \in M_2 - M_0$ and there is no $a \in M_0$ such that $y <_2 a <_1 x$.

It is straightforward to show that $<_3$ is a linear order on $M_1 \cup M_2$ and σ is an automorphism with respect to $<_3$.

We can extend $(M_1 \cup M_2, <_3)$ to some dense linear order without endpoints. By quantifier-elimination, σ is a partial elementary map in this dense linear order. Therefore, we can extend σ to some automorphism of a dense linear order without endpoint. There is a "constructive" proof also. □

Fact 3.5. *Stable theories have the PAPA.*

Proof. Let M_0, M_1, and M_2 be models of T such that $M_0 \prec M_1, M_2$, and σ_0, σ_1, σ_2 automorphisms of M_0, M_1, M_2 respectively such that $\sigma_i|M_0 = \sigma_0$ for $i = 1, 2$. We can assume that M_1 and M_2 are elementary submodels of a big model of T and they are independent over M_0.

We can also assume that σ_1 and σ_2 are automorphisms of the big model. Consider $\sigma_1^{-1}\sigma_2(M_2)$. Since $\sigma_1^{-1}\sigma_2|M_0 = id_{M_0}$, we have $\mathrm{tp}(\sigma_1^{-1}\sigma_2(M_2)/M_0) = \mathrm{tp}(M_2/M_0)$. Also, $\sigma_1^{-1}(M_1) = M_1$ as sets, $\sigma_1^{-1}\sigma_2(M_2) = \sigma_1^{-1}(M_2)$ as sets, and $\sigma_1^{-1}(M_0) = M_0$ as sets. Since $M_1 \perp_{M_0} M_2$, we have $M_1 \perp_{M_0} \sigma_1^{-1}\sigma_2(M_2)$.

Since $\mathrm{tp}(M_2/M_0)$ is stationary, $\mathrm{tp}(\sigma_1^{-1}\sigma_2(M_2)/M_1) = \mathrm{tp}(M_2/M_1)$. Therefore, there is an automorphism τ such that $\tau|(M_2 - M_0) = \sigma_1^{-1}\sigma_2|(M_2 - M_0)$ and $\tau|M_1 = id_{M_1}$. Note that $\tau|M_2 = \sigma_1^{-1}\sigma_2|M_2$. Consider automorphism $\sigma = \sigma_1\tau$. Then $\sigma|M_1 = \sigma_1|M_1$ and $\sigma|M_2 = \sigma_2|M_2$. □

Reading this proof, one might come up with the following definition and wonder whether it is equivalent to the PAPA:

Definition 3.6. A theory T has the *PAPA over the identity* if T has the PAPA assuming $\sigma_0 = id_{M_0}$ in the definition of the PAPA (Definition 3.1).

4 Theories without the PAPA

4.1 Ziegler's Example

Let $L = \{R(x, y, z), U(x)\}$. Consider $M = (\mathbb{R}, R^M, U^M)$ where $U^M = \mathbb{Q}$, $R^M(x, y, z)$ if and only if $x < y < z$ or $z < y < x$ in $\mathbb{R}$, and let $T_Z = Th(M)$.

Fact 4.1. *T_Z is ω-categorical and admits QE.*

Let $M_0 = M|(\mathbb{R} \setminus \{0\})$, $\sigma_0(x) = -x$, $M_1 = M$, $\sigma_1(x) = -x$ for $x \in \mathbb{R}$. Also, let $M_2 = (\mathbb{R}, R^{M_2}, U^{M_2})$, $R^{M_2} = R^M$, $U^{M_2} = \mathbb{Q} \setminus \{0\}$, $\sigma_2(x) = -x$ for $x \in \mathbb{R}$.

Fact 4.2. *T_Z does not have the PAPA. We cannot amalgamate (M_1, σ_1) and (M_2, σ_2) over (M_0, σ_0).*

The following shows that the PAPA and the PAPA over the identity are different.

Fact 4.3. *T_Z has the PAPA over the identity.*

Proof. In fact, after fixing two different constants, a dense linear order $<$ is interdefinable with R. We can show the PAPA adopting the proof of the PAPA for the theory of dense linear order without endpoint. □

4.2 Tsuboi's Example

Let R be a binary relation, and P a ternary relation, $G = (|G|, R^G)$ a random graph. Let $M = (|G|, P^M)$ be such that for $x, y, z \in |G|$, $P^M(x, y, z)$ iff (x, y, z) has three or no R-edges.

Then $Th(M)$ is supersimple with SU-rank 1, and does not have the PAPA. It has the PAPA over the identity. In fact, it has the PAPA after expanding by 2 distinct constants.

4.3 Ivanov's Example

Modifying the Tsuboi's example by introducing countably many relations, A. Ivanov constructed a supersimple theory with SU-rank 1 such that any expansion by finitely many constants do not have the PAPA [3]. It also have the PAPA over the identity. it has the PAPA after expanding by countably many constants.

Question 4.4. Find out the relation between the following conditions:

(1) T has the PAPA over the identity.

(2) T has the PAPA after expanding the language by constants.

In the rest of the paper, we deal with theories without PAPA in a strong sense.

Definition 4.5. A theory T has the NPAPA over the identity if for any model M_0 of T there are elementary extensions M_1, M_2 of M_0 and an automorphism σ_i of M_i with $\sigma_i|M_0$ is the identity on M_0 for each $i = 1, 2$ such that (M_1, σ_1) and (M_2, σ_2) cannot be amalgamated over (M_0, id_{M_0}).

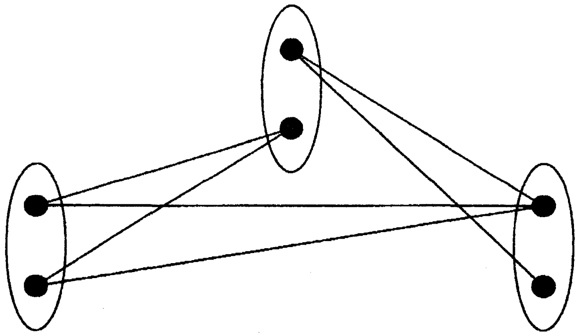
4.4 Hrushovski's Example

Let $L = \{R(x, y), E(x, y)\}$. We consider structures in which R represents the edges of a graph, and E is an equivalence relation such that each E-class has exactly 2 elements.

For any two distinct E-classes C, D, we write $C < D$ if we can write $C = \{c, c'\}$ and $D = \{d, d'\}$ with $R(c, d)$, $R(c, d')$ and there are no other R-edges among c, c', d, d'.

We consider a class $\mathbf{K}_H$ of finite $\mathcal{L}$-structures such that $A \in \mathbf{K}_H$ if and only if $A|R$ is a graph and for any two distinct E-classes C, D, either $C < D$ or $D < C$.

For example, the following is a member in $\mathbf{K}_H$:



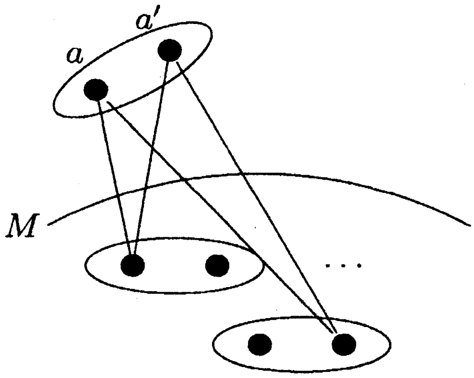
$\mathbf{K}_H$ satisfies the hypothesis of Theorem 2.2. Let T_H be the theory of the quasi-generic structure of $\mathbf{K}_H$. T_H is model complete, admits QE with definable function f swapping the elements of each E-class:

$$f(x) = y \iff E(x, y) \text{ and } x \neq y.$$

T_H is supersimple with SU-rank 1. For any model M of T_H, $M|R$ is a random graph. On the set of E-classes in M, $<$ defines a random taunament.

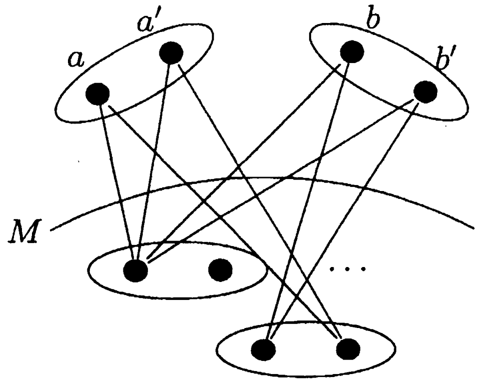
Theorem 4.6. *T_H has the NPAPA over the identity.*

Proof. We work in a big model of T_H. Let M be any model of T_H. Choose an E-class $A = \{a, a'\}$ "dominating" every E-class in M, i.e., $D < A$ for any E-class D in M.

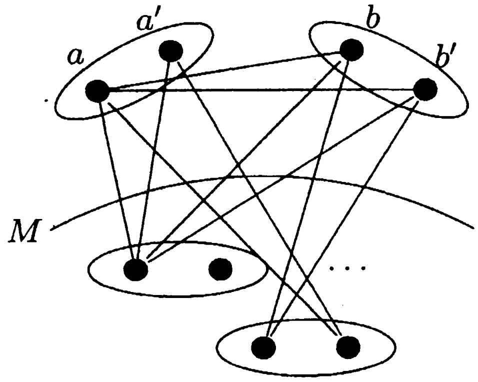


Then we have $\mathrm{tp}(a, a'/M) = \mathrm{tp}(a', a/M)$. Therefore, there is an automorphism σ_1 of the big model which fixes every element of M and swaps a and a'.

Choose another E-class $B = \{b, b'\}$ "dominating" every E-class in M such that $\mathrm{tp}(b, b'/M) \neq \mathrm{tp}(a, a'/M)$. Let σ_2 be an automorphism of the big model which fixes every element of M and swaps b and b'.



We claim that σ_1 and σ_2 cannot be amalgamated over (M, id). Suppose they are amalgamated to an automorphism σ over (M, id). We represent the amalgamated images of a, a', b, b' by the same letters. Without loss of generality, we can assume that $\{a, a'\} < \{b, b'\}$ with $R(a, b)$ and $R(a, b')$.

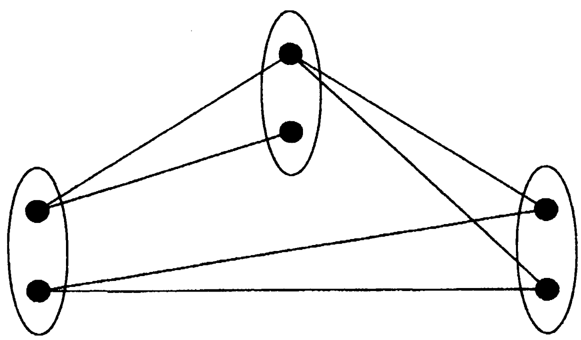


But $R(a, b)$ and $\neg R(\sigma(a), \sigma(b))$ hold. This contradicts to the assumption that σ is an automorphism. □

4.5 A Modification of Hrushovski's Example

Let $L = \{R, E\}$ be the same as one in Section 4.4 (Hrushovski's Example). Let $\mathbf{K}$ be a class of finite $\mathcal{L}$-structures such that $A \in \mathbf{K}$ if and only if $A|R$ is a finite graph, and the E-classes are linearly ordered by $<$ defined on E-classes in Section 4.4.

For example, the following is a member in $\mathbf{K}$:



Let T be the theory of the quasi-generic structure of $\mathbf{K}$. By the same argument of the proof of Theorem 4.6, we have:

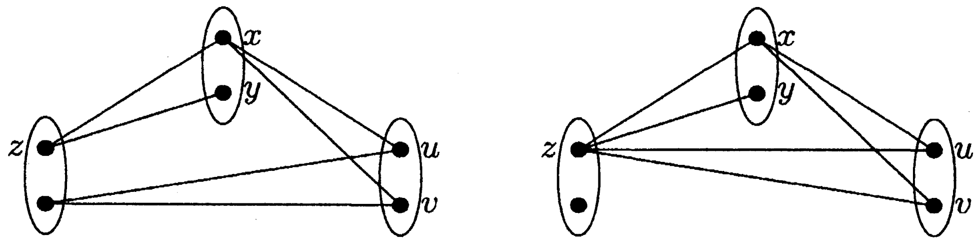
Theorem 4.7. *T has the NPAPA over the identity.*

Remark 4.8. *In T, E is definable by the following formula in $\{R\}$:*

$$E(x,y) \iff \exists u,v\,[\{u,v\}\cap\{x,y\}=\emptyset \wedge \forall z\ (R(z,x)\wedge R(z,y)\rightarrow (R(z,u)\leftrightarrow R(z,v)))]$$

Proof. We work in a model of T.

($\Rightarrow$) Suppose $E(x,y)$ holds. Then the right hand side holds by the following picture:



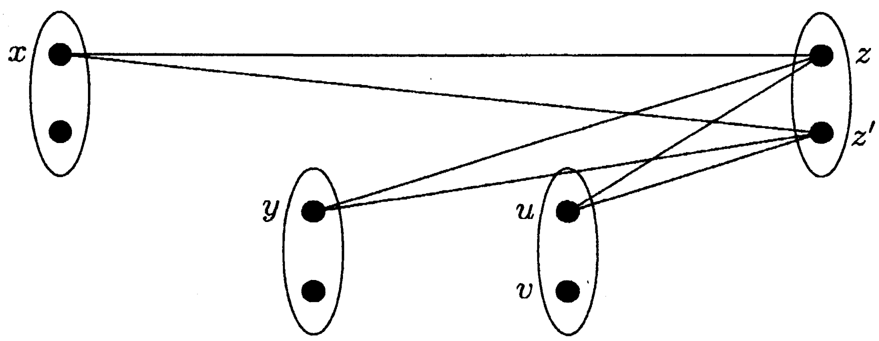
($\Leftarrow$) Suppose $\neg E(x,y)$ holds. We show the negation of the right hand side:

$$\forall u,v\,[\{u,v\}\cap\{x,y\}=\emptyset \rightarrow \exists z\ (R(z,x)\wedge R(z,y)\wedge (R(z,u)\not\leftrightarrow R(z,v)))] \qquad (1)$$

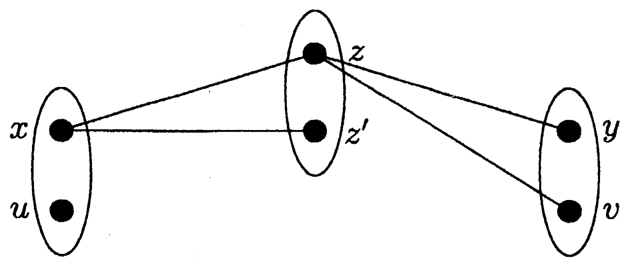
Let X and Y be the E-classes of x and y respectively. Without loss of generality, we can assume that $X<Y$.

Let u, v be arbitrary such that $\{u,v\}\cap\{x,y\}=\emptyset$.

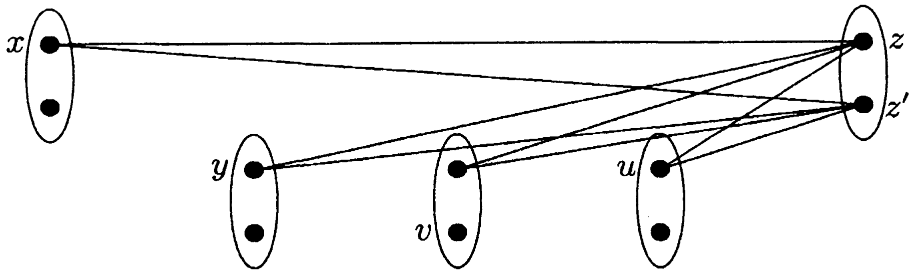
Case $E(u,v)$. We can choose an E-class $\{z,z'\}$ as the following picture, thus satisfying (1):



Case $E(x,u)$ and $E(y,v)$. We can choose an E-class $\{z,z'\}$ as the following picture, thus satisfying (1):



Otherwise, we can assume that $\neg E(x,u)$ and $\neg E(y,u)$. We can choose an E-class $\{z,z'\}$ as the following picture, thus satisfying (1):



Note that, v can be E-equivalent to x or y. $\square$

Remark 4.9. *Using a method to interpret structures in a graph described in Mekler's paper [7], from a quasi-generic class* $\mathbf{K}$ *of structures, we can construct a quasi-generic class* $\mathbf{K}'$ *of binary graphs whose quasi-generic graph has many model theoretic properties in common with the quasi-generic structure of* $\mathbf{K}$. *For example, from Hruhsovski's example, we can construct a theory* T'_H *of a graph which is supersimple with* SU*-rank 1 and has the NPAPA over the identity.*

References

[1] W. Hodges, *Model Theory*, Cambridge University Press, 1993.

[2] E. Hrushovski, H. Kikyo, and A. Tsuboi, A theory without PAPA, Kokyuroku of RIMS **1344** (2003), 11–15 (in Japanese).

[3] A. Ivanov, Automorphisms of homogeneous structures, Notre Dame J. of Formal Logic **46** (2005), No. 4, 419–424.

[4] H. Kikyo, Model companions of theories with an automorphism, J. Symbolic Logic **65** (2000), No. 3, 1215–1222.

[5] H. Kikyo, S. Shelah, The strict order property and generic automorphisms, J. Symbolic Logic **67** (2002), No. 1, 214–216.

[6] D. Lascar, Autour de la propriété du petit indice, Proc. London Math. Soc. **62** (1991) 25-53.

[7] A. Mekler, Stability of nilpotent groups of class 2 and prime exponent, J. Symbolic Logic **46** (2002), No. 4, 781–788.

[8] F. Wagner, *Simple Theories*, Kluwer Academic Publishers, Dordrecht, 2000.